**LECTURES**

**Module 1. Functions of one variable**

**Lecture 1.**

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| **The Real Numbers System. Principle of Mathematical induction.****The real number system**The real number system (which we will often call simply the reals) is first of all a set $\left\{a,b,c,…\right\}$ on which the operations of addition and multiplication are defined so that every pair of real numbers has a unique sum and product, both real numbers, with the following properties.1. $a+b=b+a$ and $a∙b=b∙a$ (*commutative laws*).
2. $\left(a+b\right)+c=a+(b+c)$ and $(a∙b)∙c=a∙(b∙c)$ *(associative laws).*
3. $a\left(b+c\right)=ab+ac$ (*distributive law*).
4. There are distinct real numbers 0 and 1 such that $a+0=a$ and $a∙1=a$ for all $a$.
5. For each $a$ there is a real number $–a$ such that $a+(-a)=0,$ and if $a\ne 0$ , there is a real number $\frac{1}{a}$ such that $a\left(\frac{1}{a}\right)=1$.
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| A set on which two operations are defined so as to have properties (1)–(5) is called *a field*. The real number system is by no means the only field. The rational numbers (which are the real numbers that can be written as $r=\frac{p}{q}$ , where $p$ and $q$ are integers and $q\ne 0)$ also form a field under addition and multiplication. The simplest possible field consists of two elements, which we denote by 0 and 1, with addition defined by  |

A set on which two operations are defined so as to have properties (A)-(E) is called a *field*.

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| 0+0=1+(-1)=0, 1+0=0+1=1 (1)  |
| and multiplication defined by$0∙0=0∙1=1∙0=0,$ $1∙1=1$ (2)**The Order Relation**The real number system is ordered by the relation <, which has the following properties.1. For each pair of real numbers $a$ and $b$, exactly one of the following is true:

$$a=b, a<b or b<a.$$1. If $a<b $and $b<c,$ then $a<c. $(the relation < is *transitive*).
2. If $a<b $and $a+c<b+c,$ for any $c$, and if $c>0$, then $ac<bc$.

A field with an order relation satisfying (6)–(8) is an *ordered field*. Thus, the real numbers form an ordered field. The rational numbers also form an ordered field, but it is impossible to define an order on the field with two elements defined by (1) and (2) so as to make it into an ordered field. **Theorem.** (The Triangle Inequality) If $a$ and $b$ are any two real numbers, then $\left|a+b\right|\leq \left|a\right|+\left|b\right|.$ (3)**Corollary.**  If $a$ and $b$ are any two real numbers, then $\left|a-b\right|\geq \left|\left|a\right|-\left|b\right|\right|,$ (4)$\left|a+b\right|\geq \left|\left|a\right|-\left|b\right|\right|.$ (5)**Definition.** A set $S$ of real numbers is *bounded above* if there is a real number $b$ such that $x\leq b$ whenever $x\in S.$**Definition.** A set $S$ of real numbers is *bounded below* if there is a real number $a$ such that $x\geq a$ whenever $x\in S.$The property of the real numbers described in the next theorem is called the Archimedean property.**Theorem.** (The Atchimedean property) If $p and ε$ are positive, then $nε>p$ for some integer $n.$  **Definition.** A set $D$ *is dence in the reals* if every open interval $(a,b)$ contains a member of $D.$  **Theorem.** The rational numbers are dence in the reals; that is if $a$ and $b$ a real numbers with $a<b,$ there is a rational number $\frac{p}{q}$ such that $a<\frac{p}{q}<b.$  |
|  **Principle of Mathematical induction.**Principle of Mathematical induction: if any proposition $P\_{n}$ is true for $n=1$, then we assume that the proposition $P\_{n}$ is true for $n=k$. If we prove that a given proposition $P\_{n}$ is true for $n=k+1,$ then $P\_{n}$ is true for each integer $n$. |
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**Example**. Let $ P\_{n}$ be the proposition that

$1+2+3+…+n=\frac{n(n+1)}{2}$ , $n\in N$ (1)

1. We are going to check identity (1) for $n=1.$ $P\_{1}$ is the proposition that , which is certainly true.
2. We assume that the proposition $P\_{n}$ is true for $n=k$. For all $n=k\in N$ we have

 $1+2+3+…+k=\frac{k(k+1)}{2}$ (2)

1. We should prove that for all $n=k+1$ the following equality is true:

$$1+2+3+…+\left(k+1\right)=\frac{(k+1)(k+2)}{2}$$

**Proof**: Adding up the first *k* + 1 integers, we have

 $1+2+3+…+k+\left(k+1\right)=\left\{we use identity (2)\right\}$=

$$=\frac{k(k+1)}{2}+\left(k+1\right)=\frac{k\left(k+1\right)+2(k+1)}{2}=\frac{(k+1)(k+2)}{2}$$

Consequently, $P\_{n}$ is true for all $n\in N.$